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# Locally trivial principal homogeneous spaces and conjugacy theorems for Lie algebras

A. Pianzola<sup>1</sup>*Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1*

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## Abstract

We link locally trivial principal homogeneous spaces over  $\text{Spec } R$  to the question of conjugacy of maximal abelian diagonalizable subalgebras of  $\mathfrak{g} \otimes R$ .

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Throughout  $k$  will denote a field of characteristic zero. Unless specifically mentioned otherwise *all algebras, tensor products, vector spaces, and schemes are over  $k$ .*

Let  $\mathfrak{g}$  be a finite dimensional split semisimple Lie algebra over  $k$ . Of central importance to classical Lie theory is Chevalley's theorem asserting the conjugacy of all split Cartan subalgebras of  $\mathfrak{g}$ . We are interested in an analogue of this result for Lie algebras of the form  $\mathfrak{g}(R) := \mathfrak{g} \otimes R$ , where  $R$  is an associative commutative unital  $k$ -algebras. (Recall that  $\mathfrak{g}(R)$  is viewed as an algebra over  $k$ . In general, these algebras are infinite dimensional.) A well understood example is the case of the algebra  $R = k[t, t^{-1}]$  of Laurent polynomials. Then  $\mathfrak{g}(R)$  is the so called loop algebra of  $\mathfrak{g}$  that one encounters on the realizations of non-twisted affine Kac–Moody Lie algebras. In this case the appropriate version of conjugacy is due to Peterson and Kac (see Remark 2(iii) below).

Let  $\mathfrak{h}$  be a split Cartan subalgebra of  $\mathfrak{g}$ . Then  $\mathfrak{h} \simeq \mathfrak{h} \otimes 1$  is *not* in general a Cartan subalgebra of  $\mathfrak{g}(R)$  (since it is not self normalized unless  $R = k$ ). The split Cartan subalgebras of  $\mathfrak{g}$  are examples of *abelian  $k$ -diagonalizable subalgebras* of  $\mathfrak{g}(R)$ , namely of subalgebras  $\mathfrak{a}$  of  $\mathfrak{g}(R)$  such that

*E-mail address:* [a.pianzola@ualberta.ca](mailto:a.pianzola@ualberta.ca).

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- (i)  $\mathfrak{a}$  is abelian;
- (ii) All elements of  $\mathfrak{a}$  are  $k$ -diagonalizable: If  $\mathfrak{p}$  belongs to  $\mathfrak{a}$  then  $\mathrm{ad}_{\mathfrak{g}(R)} \mathfrak{p}$ , when viewed as a  $k$ -linear endomorphism of  $\mathfrak{g}(R)$ , is diagonalizable.

(Any subalgebra of  $\mathfrak{g}(R)$  satisfying (ii) is abelian, but no harm is done by emphasizing this last.) If in addition

- (iii) No subalgebra of  $\mathfrak{g}(R)$  satisfying (i) and (ii) above properly contains  $\mathfrak{a}$

then  $\mathfrak{a}$  is a *maximal abelian  $k$ -diagonalizable subalgebra*, or a MAD for short. (We will see later that split Cartans of  $\mathfrak{g}$  are MADs of  $\mathfrak{g}(R)$  if and only if  $\mathrm{Spec}(R)$  is connected.)

Since these type of subalgebras play a crucial role in understanding  $\mathfrak{g}(R)$  and its representations in both the finite dimensional and affine Kac–Moody case, it is natural and relevant to ask if all MADs of  $\mathfrak{g}(R)$  are conjugate under some suitable subgroup of  $\mathrm{Aut}_{k\text{-Lie}} \mathfrak{g}(R)$ . The natural choice for this subgroup (because of functoriality on  $R$  and compatibility with the usual results in the case of a base field), is the group  $\mathfrak{G}(R)$  of  $R$ -points of the corresponding simply connected Chevalley–Demazure group, acting on  $\mathfrak{g}(R)$  via the adjoint representation. As we shall see, the answer to this question is quite interesting and related to the triviality of certain principal homogeneous spaces over  $\mathrm{Spec}(R)$ .

Again by analogy with the finite dimensional case, one expects regular elements to play a special role in the problem at hand. The correct functorial definition for these elements is as follows.<sup>2</sup> Let  $f_{\mathrm{reg}} \in S(\mathfrak{g}^*)$  be the polynomial function defining the basic Zariski open dense set of regular elements of  $\mathfrak{g}$  (see [2, Chapter VII]). Since  $f_{\mathrm{reg}}$  is defined over  $k$ , we can think of it as a polynomial function on the free  $R$ -module  $\mathfrak{g}(R)$ . An element  $\mathfrak{p}$  of  $\mathfrak{g}(R)$  will be said to be *regular* if  $f_{\mathrm{reg}}(\mathfrak{p})$  is a unit of  $R$ . Finally, a MAD is said to be *regular* if it contains a regular element.

Here then is our main result.

**Theorem 1.** *Let  $\mathfrak{g}$  be a finite dimensional split semisimple Lie algebra over  $k$ , and  $\mathfrak{G}$  its simply connected Chevalley–Demazure group scheme. Let  $\mathfrak{X} = \mathrm{Spec}(R)$  be a connected affine scheme and  $\mathfrak{X}_{\mathrm{red}}$  the corresponding reduced scheme. Then*

- (i) *If  $\mathfrak{a}$  is an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{g}(R)$  then  $\dim_k(\mathfrak{a}) \leq \mathrm{rank}(\mathfrak{g})$ . If this is an equality then  $\mathfrak{a}$  is maximal.*
- (ii) *Assume that  $\mathfrak{X}(k) \neq \emptyset$ .*
  - (a) *(Regular conjugacy.) If the Picard group of  $\mathfrak{X}_{\mathrm{red}}$  is trivial then all regular maximal abelian  $k$ -diagonalizable subalgebras of  $\mathfrak{g}(R)$  are conjugate under  $\mathfrak{G}(R)$ .*
  - (b) *(Full conjugacy.) Consider the following property on  $\mathfrak{X}$ .*

<sup>2</sup> J.-P. Serre suggested this definition. See also Exposé XIV of [18].

(TLT) (Triviality of locally trivial Levi torsors.) *If  $\mathfrak{L}$  is the Levi subgroup of a standard parabolic subgroup of  $\mathfrak{G}$ , then any locally trivial principal homogeneous space for  $\mathfrak{L}$  over  $\mathfrak{X}_{\text{red}}$  is trivial.*

*If (TLT) holds, then all maximal abelian  $k$ -diagonalizable subalgebras of  $\mathfrak{g}(R)$  are regular (and hence all conjugate by (a)).*

The main idea behind the proof of Theorem 1 is to evaluate the different primes of  $\mathfrak{X}$  at a given  $k$ -diagonalizable element of  $\mathfrak{g}(R)$ . Each of these evaluations puts us in the finite dimensional case where conjugacy is known to hold. One then is forced to look at assumptions on  $\mathfrak{X}$  that allow all of these finite dimensional conjugacies to be “pasted together” to create an element of  $\mathfrak{G}(R)$ . The proof of the first part of Theorem 1 is straightforward and is given earlier in the paper after developing some basic properties of  $k$ -diagonalizable elements. This is followed by a series of results that conclude in Proposition 11 with the translation of the conjugacy question to one on the triviality of certain torsors (= principal homogeneous spaces. See Remark 2(ii) below) over  $\text{Spec}(R)$ . An induction argument is then used to prove the second part of the main theorem. The paper concludes with an interesting example.

## 2. Remarks

(i) There exist rings  $R$  leading to non regular MADs, and if  $\text{Pic}(\mathfrak{X}) \neq 0$  regular MADs need not be conjugate. The connected assumption on  $\mathfrak{X}$  is needed in part (i) of the theorem but is not crucial for part (ii) (which holds if  $\mathfrak{X}$  has a finite number of connected components each of which satisfies the assumptions of the theorem). On the other hand the assumption on the existence of a rational point, namely of a maximal ideal  $x_0$  such that  $R/x_0 \simeq k$ , is central to the proof.

(ii) (See [9] and [6] for details.) Let  $\mathfrak{X}$  be a  $k$ -scheme, and let  $\mathfrak{L}$  be an algebraic  $k$ -group. A (right)  $\mathfrak{L}$ -torsor over  $\mathfrak{X}$  (also called an  $\mathfrak{X}$ -torsor under  $\mathfrak{L}$ ) is an  $\mathfrak{X}$ -scheme  $\mathfrak{Y}$  on which  $\mathfrak{L}_{\mathfrak{X}} := \mathfrak{L} \times_{\text{Spec}(k)} \mathfrak{X}$  acts on the right, and which is locally isomorphic to  $\mathfrak{L}_{\mathfrak{X}}$  for the flat topology of  $\mathfrak{X}$  (with  $\mathfrak{L}_{\mathfrak{X}}$  acting on itself by right multiplication.) Thus there exist flat and locally finitely presented morphisms  $\phi_i: \mathcal{U}_i \rightarrow \mathfrak{X}$  with  $\mathfrak{X} = \bigcup \phi_i(\mathcal{U}_i)$  and  $\mathfrak{Y} \times_{\mathfrak{X}} \mathcal{U}_i \simeq \mathfrak{L}_{\mathcal{U}_i} := \mathfrak{L}_{\mathfrak{X}} \times_{\mathfrak{X}} \mathcal{U}_i$  (these isomorphisms preserving the respective  $\mathfrak{L}_{\mathcal{U}_i}$  actions). If our group is smooth the  $\phi_i$  may be taken to be étale, and then just as with principal bundles in differential geometry (of which torsors are a suitable algebraic analogues) we can attach to the isomorphism class of a torsor  $\mathfrak{Y}$  as above an element of  $H_{\text{ét}}^1(\mathfrak{X}, \mathfrak{L}_{\mathfrak{X}})$  (Čech cohomology on the étale site of  $\mathfrak{X}$  with coefficients on the group sheaf  $\mathfrak{L}_{\mathfrak{X}}$ ). This is an injective procedure, and it is surjective if  $\mathfrak{X}$  is affine.  $H_{\text{ét}}^1(\mathfrak{X}, \mathfrak{L}_{\mathfrak{X}})$  is a set with a distinguished element, namely the isomorphisms class of the trivial torsor  $\mathfrak{L}_{\mathfrak{X}}$  acting on itself by right multiplication. If the  $\phi_i$  can be taken to be open immersions (the  $\mathcal{U}_i$  can then be thought as an honest open cover of  $\mathfrak{X}$  in the Zariski topology), the torsor  $\mathfrak{Y}$  is said to be *locally trivial*. Their isomorphism classes are then parametrized (again assuming  $\mathfrak{X}$  affine and  $\mathfrak{L}$  smooth) by  $H_{\text{Zar}}^1(\mathfrak{X}, \mathfrak{L}_{\mathfrak{X}})$ .

(iii) Condition (TLT) varies with the type of  $\mathfrak{g}$  and the nature of the base field  $k$ . For the condition to hold for all types it suffices to assume that  $\mathfrak{X}$  has the following property

(TRT) (Triviality of locally trivial reductive torsors.) *If  $\mathcal{L}$  is a (connected) split reductive  $k$ -group then any locally trivial principal homogeneous space for  $\mathcal{L}$  over  $\mathfrak{X}_{\text{red}}$  is trivial.*

There are two important examples of rings with this property, namely those  $R$  which modulo their nilradical equal either

- (a)  $k[t_1, \dots, t_n]$ , or
- (b)  $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ .

Case (a) follows from the work of Raghunathan and Ramanathan [17], and of Raghunathan dealing with the triviality of certain torsors over algebraic affine space ([15, Theorem 2.2], see also [16] and [4]). Case (b) reduces to case (a). For  $n = 1$  this is easy since every locally trivial principal homogeneous space under  $\mathcal{G}$  over the punctured affine line, extends to one over the whole affine line. (In fact because our torsors are locally trivial, one can directly show that (TLT) holds for  $\mathfrak{X} = \text{Spec}(k[t^{\pm 1}])$  by means of a standard argument. See, for example, the proof of Proposition 3(i) of [13].) Note that this recovers the conjugacy theorem of Peterson–Kac in the case of untwisted affine Kac–Moody Lie algebras (see [11]). The general case follows by an induction argument due to Gille [7]. Case (b) is of great importance because of toroidal Lie algebras (see [12] and [14]).

The proof of the theorem is slightly easier if one assumes (TRT) rather than (TLT) (see the Remark following Proposition 9). Non standard examples where (TLT) holds can be found with the aid of Théorème 6.13 of [5].

(iv) Note that we are dealing with the triviality of certain *algebraic* principal bundles over the *global* space  $\mathfrak{X} = \text{Spec}(R)$ . In particular one is not allowed to replace  $R$  by any of its localizations or completions (indeed, conjugacy may hold for all localization of  $R$  yet fail for  $R$  itself). That we are in the algebraic setup forces us to work, even if the base field  $k$  is  $\mathbb{R}$  or  $\mathbb{C}$ , with the Zariski topology and the complications that this entails for fibrations (compare, for example, the triviality of vector bundles over affine space in the classical case by contractibility, with its “Serre’s conjecture” algebraic counterpart: Theorems of Quillen and Suslin). The work of Raghunathan is here crucial.

It is important to observe that though MADs behave somehow functorially on  $R$  (Lemma 5 and Proposition 6), MADs are not  $R$ -modules. The point is that  $k$ -diagonalizability is lost in general by scaling under elements of  $R$  which are not in  $k$ . In fact the role of  $k$ -diagonalizability is crucial but deceptively subtle and may at times be easily overlooked.

### 3. Notation and conventions

Throughout  $\mathfrak{g}$  and  $\mathcal{G}$  will be as in the statement of Theorem 1. The category of commutative associative unital  $k$ -algebras will be denoted by  $k\text{-alg}$ . If  $R$  is in  $k\text{-alg}$  the residue field of an element  $x$  of  $\text{Spec}(R) = \mathfrak{X}$  will be denoted by  $k(x)$ . For convenience in

what follows the group  $\mathfrak{G}(k(x))$  will be denoted simply by  $\mathfrak{G}(x)$ , and the corresponding group homomorphism  $\mathfrak{G}(R) \rightarrow \mathfrak{G}(R/x) \subset \mathfrak{G}(x)$  by  $\mathfrak{p} \mapsto \mathfrak{p}(x)$ .

The constructions of the last paragraph can be repeated, *mutatis mutandi*, if we replace  $\mathfrak{G}$  by its Lie algebra functor  $\mathfrak{g}(-)$ . Since  $\mathfrak{g}$  is finite dimensional we have  $\mathfrak{g}(-) = \text{Hom}_{k\text{-alg}}(S(\mathfrak{g}^*), -)$ . Thus  $\mathfrak{g}(S) = \mathfrak{g} \otimes S$  for any  $S$  in  $k\text{-alg}$ . In particular,  $\mathfrak{g} \simeq \mathfrak{g}(k)$ .

Along similar lines if  $V$  is a vector space over  $k$ ,  $S$  is in  $k\text{-alg}$ , and  $x \in \mathfrak{X}$ ; we will denote  $V \otimes S$  by  $V(S)$  and  $V(k(x))$  by  $V(x)$ .

Let  $k[\mathfrak{G}]$  be the coordinate ring of  $\mathfrak{G}$ . There is then a dual nature to  $\mathfrak{G}$ . It can be thought as the scheme  $\text{Spec}(k[\mathfrak{G}])$  or as the functor  $\text{Hom}_k(k[\mathfrak{G}], -)$  from  $k\text{-alg}$  into the category of groups. We shall make use of both these manifestations. The following example may help clarify these ideas.

**Example.** Take  $\mathfrak{g}$  to be of type  $\mathfrak{sl}_n$ ,  $k = \mathbb{R}$ , and  $R = \mathbb{R}[t, t^{-1}]$ . Then  $\mathfrak{G} = \mathbf{SL}_n = \text{Hom}_{\mathbb{R}\text{-alg}}(\mathbb{R}[\mathfrak{G}], -)$  with  $\mathbb{R}[\mathfrak{G}] = \mathbb{R}[x_{ij}]/(\det - 1)$ .  $\mathfrak{G}(R)$  (respectively,  $\mathfrak{g}(R)$ ) is the group (Lie algebra) of  $n \times n$  matrices of determinant 1 (trace 0) with entries in the ring  $R$ . We have  $\mathfrak{X} = \{\langle f(t) \rangle, \langle g(t) \rangle, \{0\}\}$  where  $f(t)$  and  $g(t)$  are irreducible of degree 1 and 2, respectively, and  $f(0) \neq 0$ . The corresponding residue fields of these three types of primes are isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{R}(t)$ , respectively. If  $\mathfrak{p}$  is an element of  $\mathfrak{G}(R)$  and  $x \in \mathfrak{X}$ , then  $\mathfrak{p}(x)$  is simply the matrix obtained by reducing mod  $x$  each of the entries of  $\mathfrak{p}$ . Similarly for  $\mathfrak{p} \in \mathfrak{g}(R)$ .

We begin by recalling an important fact of which we will make repeated use in what follows.

**Proposition 4.** *Let  $\mathfrak{a}$  be an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{g}(R)$ . Assume  $R$  is an integral domain, and let  $K$  denote its field of quotients. Then there exists a split Cartan subalgebra  $\mathfrak{k}$  of the  $K$ -Lie algebra  $\mathfrak{g}(K)$  with  $\mathfrak{a} \subset \mathfrak{k}$ .*

**Proof.** See Seligman [19]. See also [2, Chapter 8, Exercise §3.10(b)].  $\square$

The following result is straightforward.

**Lemma 5.** *Let  $x \in \mathfrak{X}$ .*

- (i) *If  $\mathfrak{a}$  is an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{g}(R)$  then  $\mathfrak{a}(x) := \{\mathfrak{p}(x) : \mathfrak{p} \in \mathfrak{a}\}$  is abelian and  $k$ -diagonalizable when viewed as a subalgebra of either  $\mathfrak{g}(R/x)$  or  $\mathfrak{g}(x)$ . In particular, if  $\mathfrak{p}$  is  $k$ -diagonalizable then so is  $\mathfrak{p}(x)$ .*
- (ii) *If  $\mathfrak{p} \in \mathfrak{g}(R)$  is regular then  $\mathfrak{p}(x)$  is regular.*  $\square$

Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a finite dimensional representation of  $\mathfrak{g}$ . For any  $S$  in  $k\text{-alg}$  we then get a representation  $\rho_S : \mathfrak{g}(S) \rightarrow \mathfrak{gl}(V(S))$  of the  $S$ -Lie algebra  $\mathfrak{g}(S)$ . If  $\mathfrak{p} \in \mathfrak{g}(S)$  then  $\rho_S(\mathfrak{p})^m$  is an  $S$ -linear endomorphism of the free  $S$ -module of finite rank  $V(S)$ . It is meaningful therefore to consider its trace.

**Proposition 6.** Let  $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a finite dimensional representation of  $\mathfrak{g}$ . Assume  $\mathfrak{X}$  is connected and reduced. If  $\mathfrak{p} \in \mathfrak{g}(R)$  is  $k$ -diagonalizable then

- (i)  $\text{Tr } \rho_R(\mathfrak{p})^m \in k$  for all  $m \in \mathbb{N}$ .
- (ii) If  $\mathfrak{p}(y) = 0$  for some  $y \in \mathfrak{X}$  then  $\mathfrak{p} = 0$ .

**Proof.** We reason in stages.

Step 1: *Reduction to the noetherian case.* Let  $R'$  be a finitely generated subalgebra of  $R$  such that  $\mathfrak{p}$  can be viewed as an element  $\mathfrak{p}'$  of  $\mathfrak{g}(R') \subset \mathfrak{g}(R)$ . Clearly  $\mathfrak{X}' = \text{Spec}(R')$  is connected and reduced, and  $\mathfrak{p}'$  is a  $k$ -diagonalizable element of  $\mathfrak{g}(R')$ . Since  $\text{Tr } \rho_{R'}(\mathfrak{p}')^m = \text{Tr } \rho_R(\mathfrak{p})^m$  and  $\mathfrak{p}(y) = 0$  implies  $\mathfrak{p}'(y \cap R') = 0$ , it will suffice to establish the result under the assumption that  $R$  is noetherian.

Step 2:  *$\mathfrak{X}$  integral.* Let  $K$  be the field of quotients of  $R$ . Since  $\text{ad}_{\mathfrak{g}(K)} \mathfrak{p}$  is semisimple  $\rho_K(\mathfrak{p})$  acts semisimply on  $V(K)$ . We claim that the eigenvalues of  $\rho_K(\mathfrak{p})$  (in the algebraic closure of  $K$ ) belong to  $k$ . To see this put  $\mathfrak{p}$  inside a split Cartan  $\mathfrak{k}$  of  $\mathfrak{g}(K)$  as in Proposition 4, and fix a base  $\Pi = \{\alpha_1 \dots \alpha_\ell\}$  of the corresponding root system  $\Delta = \Delta(\mathfrak{g}(K), \mathfrak{k})$ . If  $\omega_1^\vee, \dots, \omega_\ell^\vee$  are the fundamental coweights corresponding to the  $\alpha_i$ 's then  $\mathfrak{k} = \bigoplus K \omega_i^\vee$ . Now if we write  $\mathfrak{p} = \sum c_i \omega_i^\vee$  then the  $c_i$ 's are eigenvalues of  $\text{ad}_{\mathfrak{g}(K)} \mathfrak{p}$  and therefore belong to  $k$ . Since the weight space decomposition with respect to  $\mathfrak{k}$  of the  $\mathfrak{g}(K)$ -module afforded by  $\rho_K(\mathfrak{p})$  are rational linear combinations of the  $\alpha_i$ 's the claim follows. From the above we conclude that  $\text{Tr } \rho_K(\mathfrak{p})^m := \lambda_m \in k$  for all  $m \in \mathbb{N}$ . Since  $\rho_R(\mathfrak{p})$  is the restriction of  $\rho_K(\mathfrak{p})$  to  $V(R)$  (i) holds.

If  $\mathfrak{p}(y) = 0$  then  $\lambda_m + y = (\text{Tr } \rho_R(\mathfrak{p})^m) + y = \text{Tr } \rho_{R/y}(\mathfrak{p}(y))^m = 0$  so  $\lambda_m = 0$ . Thus  $\text{Tr } \rho_K(\mathfrak{p})^m = 0$  for all  $m$  and therefore  $\rho_K(\mathfrak{p}) = 0$ . Applying this to the adjoint representation yields that  $\mathfrak{p} = 0$ . This finishes the proof in the integral domain case.

Step 3:  *$\mathfrak{X}$  connected and reduced.* For  $x \in \mathfrak{X}$  we view  $\mathfrak{p}(x)$  as a  $k$ -diagonalizable element of  $\mathfrak{g}(R/x)$  (see last lemma). Fix  $m \in \mathbb{N}$  and let  $r = \text{Tr } \rho_R(\mathfrak{p})^m \in R$  and  $r_x = \text{Tr } \rho_{R/x}(\mathfrak{p}(x))^m \in R/x$ . The integral domain case yields that  $r_x \in k \subset R/x$ . Clearly  $r_y = r_x + y$  whenever  $x \subset y$ , and therefore  $r_x \in k$  is constant on the irreducible components of  $\mathfrak{X}$  (take  $x$  to be a minimal ideal), hence constant on  $\mathfrak{X}$  (by [8, Chapter 0, Corollary 2.1.10] since  $\mathfrak{X}$  is connected and may be assumed noetherian). Call this common value  $\lambda$ . Since  $\lambda = r_x = r + x$  for all  $x$  and  $R$  is reduced it follows that  $r = \lambda$ , hence that  $r \in k$ .

Finally assume  $\mathfrak{p}(y) = 0$ . Let  $x \subset y$  be a minimal prime, and view  $\mathfrak{p}(x)$  as a  $k$ -diagonalizable element of  $\mathfrak{g}(R/x)$  and  $y$  as an element of  $\text{Spec}(R/x)$ . Since  $\mathfrak{p}(x)(y) = \mathfrak{p}(y) = 0$ , the integral domain case yields that  $\mathfrak{p}(x) = 0$ . Thus  $\mathfrak{p}$  vanishes in the irreducible component corresponding to  $x$  and hence everywhere in  $\mathfrak{X}$  as we saw above. Since  $R$  is reduced this forces  $\mathfrak{p} = 0$ .  $\square$

**Remark.** For a given  $\lambda \in k$  let  $\mathfrak{g}(\mathfrak{p})^\lambda = \{v \in \mathfrak{g}(R): [\mathfrak{p}, v] = \lambda v\}$ . Then  $\mathfrak{g}(\mathfrak{p})^\lambda$  is a projective  $R$ -submodule of  $\mathfrak{g}(R)$ . Conjugacy is related to the freeness of these submodules.

**Proof of Theorem 1(i).** Let  $\mathfrak{a}$  be an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{g}(R)$ . Assume first that in addition of being connected  $\mathfrak{X}$  is reduced. Then by Lemma 5(i) and Proposition 6(ii) elements of  $\mathfrak{a}$  are linearly independent if and only if they are so after evaluation at any element of  $\mathfrak{X}$ . It follows that it will suffice to establish our result under the

assumption that  $R$  is an integral domain. Let then  $\mathfrak{a} \subset \mathfrak{k}$  and  $K$  be as in Proposition 4, and fix a base  $\Pi = \{\alpha_1 \dots \alpha_\ell\}$  of the corresponding root system  $\Delta = \Delta(\mathfrak{g}(K), \mathfrak{k})$ . If  $\omega_1^\vee, \dots, \omega_\ell^\vee$  are the fundamental coweights corresponding to the  $\alpha_i$ 's then by reasoning as in Step 2 of the last proposition we conclude that  $\mathfrak{a} \subset \bigoplus k\omega_i^\vee$ . This finishes the proof in the reduced case.

In general let  $\mathcal{J}$  be the nilradical of  $R$  and set  $R' = R/\mathcal{J}$ . It is clear that the image  $\mathfrak{a}'$  of  $\mathfrak{a}$  under the canonical map  $\mathfrak{g}(R) \rightarrow \mathfrak{g}(R')$  is abelian and  $k$ -diagonalizable. It then follows from the reduced case that for any given elements  $\{p_1 \dots p_{\ell+1}\}$  of  $\mathfrak{a}$ , we can find a nontrivial linear dependence relation  $c_1 p'_1 + \dots + c_{\ell+1} p'_{\ell+1} = 0$  (where, of course, the  $c'_i$ 's depend on the elements, and  $p'_i$  denotes the image of  $p_i$  under the canonical map). Consider now the element  $p := c_1 p_1 + \dots + c_{\ell+1} p_{\ell+1} \in \mathfrak{a}$ . Then the coordinates of  $p$  with respect to any basis of  $\mathfrak{g}$  (viewed as an  $R$ -basis of  $\mathfrak{g}(R)$ ) are in  $\mathcal{J}$ . From this it follows that  $\text{ad}_{\mathfrak{g}(R)} p$  is nilpotent. On the other hand since  $p \in \mathfrak{a}$  we also have that  $\text{ad}_{\mathfrak{g}(R)} p$  is diagonalizable (as a  $k$ -linear endomorphism of  $\mathfrak{g}(R)$ ). It follows that  $\text{ad}_{\mathfrak{g}(R)} p = 0$  and hence that  $p = 0$  since  $\mathfrak{g}(R)$  has trivial centre.  $\square$

**Corollary.** *Let  $\mathfrak{h}$  be a split Cartan subalgebra of  $\mathfrak{g}$ . Assume  $\mathfrak{X} = \text{Spec}(R)$  is connected. Then  $\mathfrak{h}$  is the unique MAD of  $\mathfrak{g}(R)$  contained in  $\mathfrak{h}(R)$ .*

**Proof.** Clearly  $\mathfrak{h} \subset \mathfrak{g}(R)$  is abelian and  $k$ -diagonalizable, hence maximal because of its dimension. If  $\mathfrak{k} \subset \mathfrak{h}(R)$  is an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{g}(R)$ , then  $\mathfrak{h} + \mathfrak{k}$  is also abelian and  $k$ -diagonalizable. Again a dimension argument shows that  $\mathfrak{k} \subset \mathfrak{h}$ .  $\square$

**Remark.** One can also give a direct proof of this corollary. Note that the connectedness assumption is also necessary. Indeed if  $h \in \mathfrak{h}$  and  $e \in R$  is an idempotent, then  $h \otimes e$  is a  $k$ -diagonalizable element of  $\mathfrak{g}(R)$  commuting with  $\mathfrak{h}$ .

The next result is crucial. Its effect is that the structure groups of the torsors related to conjugacy are *connected*.

**Proposition 7.** *Let  $\mathfrak{X} = \text{Spec}(R)$  be connected reduced and with a rational point. Let  $p \in \mathfrak{g}(R)$  be  $k$ -diagonalizable. Fix  $x_0 \in \mathfrak{X}$  such that  $k(x_0) = k$  and set  $p_0 := p(x_0)$ . If  $x \in \mathfrak{X}$  then  $p(x)$  and  $p_0$  (viewed as two elements of  $\mathfrak{g}(x)$ ) are conjugate under  $\mathfrak{G}(x)$ .*

**Proof.** By Lemma 5(i) and Proposition 4 (with  $R = k$ ) there exists a split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  with  $p_0 \in \mathfrak{h}$ . Now  $\mathfrak{h}(x)$  is a split Cartan subalgebra of  $\mathfrak{g}(x)$  and since any two such are conjugate under  $\mathfrak{G}(x)$  [2, Chapter 8, §3.3, Corollary to Proposition 10] there is no loss of generality in assuming that both  $p(x)$  and  $p_0$  belong to  $\mathfrak{h}(x)$ .

Under this assumption, were  $p_0$  and  $p(x)$  not conjugate under  $\mathfrak{G}(x)$ , they would be separated by a polynomial function  $f \in S(\mathfrak{h}(x)^*)$  which is invariant under the Weyl group  $W$  of  $(\mathfrak{g}(x), \mathfrak{h}(x))$  (ibid., Remarqué, §5.2, and §8.4, Lemma 6). Now any such  $f$  is a linear combination of functions of the form  $z \mapsto \text{Tr } \rho_{k(x)}(z)^m$  with  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  a finite

dimensional representation of  $\mathfrak{g}$  (ibid., §8.2, Corollary 2). But this is not possible. Indeed, functoriality combined with Proposition 6(i) yields

$$\mathrm{Tr} \rho_{k(x)}(\mathfrak{p}(x))^m = (\mathrm{Tr} \rho_R(\mathfrak{p})^m)(x) = (\mathrm{Tr} \rho_R(\mathfrak{p})^m)(x_0) = \mathrm{Tr} \rho_{k(x)}(\mathfrak{p}_0)^m. \quad \square$$

**Proposition 8.** *Let  $\mathfrak{p}_0$  be a  $k$ -diagonalizable element of  $\mathfrak{g}$ . Then*

- (i)  $\mathfrak{G}(k) \cdot \mathfrak{p}_0 \subset \mathfrak{g}$  is a Zariski closed set.
- (ii) Let  $J \triangleleft S(\mathfrak{g}^*)$  be the defining ideal of the affine variety  $\mathfrak{G}(k) \cdot \mathfrak{p}_0$ . Then the elements of  $J$  vanish on  $\mathfrak{G}(S) \cdot \mathfrak{p}_0$  for any  $k$ -algebra  $S$ .

**Proof.** (i) Let  $\bar{k}$  be the algebraic closure of  $k$ . By [3, Theorem 9.2(ii)]  $\mathfrak{G}(\bar{k}) \cdot \mathfrak{p}_0$  is a closed subset of  $\mathfrak{g}(\bar{k})$ . Since the Zariski topology of  $\mathfrak{g}(\bar{k})$  induces that of  $\mathfrak{g}$  it will suffice to show that

$$\mathfrak{G}(\bar{k}) \cdot \mathfrak{p}_0 \cap \mathfrak{g} = \mathfrak{G}(k) \cdot \mathfrak{p}_0. \quad (8.1)$$

Let  $\mathfrak{q} = \mathfrak{p} \cdot \mathfrak{p}_0 \in \mathfrak{g}$  for some  $\mathfrak{p} \in \mathfrak{G}(\bar{k})$ . It is easy to see that  $\mathfrak{q}$  is  $k$ -diagonalizable. We now use Proposition 4 (again with  $R = k$ ) and conjugacy of split Cartans to see that to establish (8.1), we may assume that  $\mathfrak{q} \in \mathfrak{h}$  where  $\mathfrak{h}$  is some fixed split Cartan containing  $\mathfrak{p}_0$ . We then have two elements  $\mathfrak{p}_0$  and  $\mathfrak{q}$  of  $\mathfrak{h}$  which are conjugate under  $\mathfrak{G}(\bar{k})$ . A standard argument using the Bruhat decomposition of  $\mathfrak{G}(\bar{k})$  shows that  $\mathfrak{q} = w(\mathfrak{p}_0)$  for some  $w$  in the Weyl group  $W$  of  $\Delta(\mathfrak{g}, \mathfrak{h})$ . Since  $w$  is the restriction to  $\mathfrak{h}$  of an element of  $\mathfrak{G}(k)$  [2, Chapter 8, §5, 3 Remarqué] (8.1) holds. This finishes the proof of (i).

(ii) The defining ideal of  $\mathfrak{G}(k) \cdot \mathfrak{p}_0$  is  $J := \{f \in S(\mathfrak{g}^*) : f \text{ vanishes in } \mathfrak{G}(k) \cdot \mathfrak{p}_0\}$ . First we assume that  $S$  is an integral domain. In this case we establish (ii) by showing below that  $E := \mathfrak{G}(k) \cdot \mathfrak{p}_0$  is dense on  $\mathfrak{G}(F) \cdot \mathfrak{p}_0$  where  $F$  is the algebraic closure of the quotient field of  $S$ .

Let  $T(E, \bar{E}) = \{\mathfrak{p} \in \mathfrak{G}(F) : \mathfrak{p} \cdot E \subset \bar{E}\}$  (hereafter  $\bar{\phantom{x}}$  denotes Zariski closure). This is a closed subset of the affine variety  $\mathfrak{G}(F) \cdot \mathfrak{p}_0 \subset \mathfrak{g}(F)$  [3, Proposition 1.7(ii)]. Since  $\mathfrak{G}(k)$  is dense in  $\mathfrak{G}(F)$  (ibid., 18.3) we obtain

$$\mathfrak{G}(k) \subset T(E, \bar{E}) \implies \overline{\mathfrak{G}(k)} \subset \overline{T(E, \bar{E})} = T(E, \bar{E}) \implies \mathfrak{G}(F) \subset T(E, \bar{E}).$$

In other words:  $T(E, \bar{E}) = \mathfrak{G}(F)$ . It follows then that in  $\mathfrak{g}(F)$  we have

$$\mathfrak{G}(F) \cdot \mathfrak{p}_0 = \mathfrak{G}(F) \cdot \mathfrak{G}(k) \cdot \mathfrak{p}_0 = \mathfrak{G}(F) \cdot E \subset \bar{E} = \overline{\mathfrak{G}(k) \cdot \mathfrak{p}_0} \subset \overline{\mathfrak{G}(F) \cdot \mathfrak{p}_0} = \mathfrak{G}(F) \cdot \mathfrak{p}_0.$$

Thus  $\mathfrak{G}(F) \cdot \mathfrak{p}_0 = \overline{\mathfrak{G}(k) \cdot \mathfrak{p}_0}$  as desired. This establishes our result for integral domains.

Let  $S$  now be arbitrary. Given that  $k[\mathfrak{G}]$  is an integral domain the elements of  $J$  vanish at  $\mathrm{id} \cdot \mathfrak{p}_0$  where  $\mathrm{id} \in \mathfrak{G}(k[\mathfrak{G}])$ . Since any element of  $\mathfrak{G}(S)$  is of the form  $\phi(\mathrm{id})$  for some arrow  $\phi : k[\mathfrak{G}] \rightarrow S$  the results holds for  $S$  by functoriality.  $\square$

**Proposition 9.** *Let  $\mathfrak{p}_0$  be a  $k$ -diagonalizable element of  $\mathfrak{g}$ , and let  $\mathcal{L}$  be its isotropy group (i.e.,  $\mathcal{L}(S) = \{\mathfrak{p} \in \mathfrak{G}(S) : \mathfrak{p} \cdot \mathfrak{p}_0 = \mathfrak{p}_0\}$  for any  $S$  in  $k$ -alg). Let  $\mathfrak{h}$  be a fixed split Cartan*



subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{p}_0$ . Then there exists a base  $\Pi = \{\alpha_1 \dots \alpha_\ell\}$  of  $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$  and a subset  $I \subset \{1, \dots, \ell\}$  such that

- (i)  $\mathcal{L}$  is the standard Levi subgroup corresponding to  $I$ . In particular,  $\mathcal{L}$  is a split (connected) reductive algebraic group.
- (ii) The derived group  $\mathfrak{G}_I$  of  $\mathcal{L}$  is of simply connected type.
- (iii) If condition (TLT) on  $\mathfrak{X}$  holds for  $\mathfrak{G}$ , then it also holds for  $\mathfrak{G}_I$ .

**Proof.** Let  $\Delta(\mathfrak{p}_0) = \{\alpha \in \Delta: \langle \alpha, \mathfrak{p}_0 \rangle = 0\}$ . If  $\Delta(\mathfrak{p}_0) = \emptyset$  set  $I = \{1, \dots, \ell\}$ . If not, then  $\Delta(\mathfrak{p}_0)$  is a root system on the subspace of  $\mathfrak{h}^*$  it spans, and there exists a subset  $I = I(\mathfrak{p}_0) \subset \{1, \dots, \ell\}$  such that  $\Pi_I := \{\alpha_i: i \in I\}$  is a base of  $\Delta(\mathfrak{p}_0)$  [1, Chapter VI, Proposition 24].

(i) Over  $\bar{k}$  this follows from Lemma 3.7 and Corollary 3.11 of [20]. For the general case one has to check that all relevant arguments hold over  $k$  (e.g., the Bruhat decomposition of  $\mathfrak{G}(k)$ ).

(ii) By [18, Expose XXII] the derived group of  $\mathcal{L}$  is generated (as a sheaf group on the flat site of  $\mathfrak{X}$ ) by the root subgroups corresponding to roots whose support lies in  $\Pi_I$ . We denote this group by  $\mathfrak{G}_I$ . That  $\mathfrak{G}_I$  is simply connected means that the geometric fibers of  $\mathfrak{G}_I$  are simply connected algebraic groups, which holds by [20, 2.11 and 2.13] (see also [21, Chapter 2, Corollary 5.4]).

(iii) We must show that  $H_{\text{Zar}}^1(\mathfrak{X}', \mathcal{L}') = (0)$  where  $\mathfrak{X}' = \mathfrak{X}_{\text{red}}$ , and  $\mathcal{L}'$  is the standard Levi subgroup corresponding to a subset  $I'$  of  $I$ . (Strictly speaking our group is  $\mathcal{L}'_{\mathfrak{X}} := \mathcal{L}' \times_{\text{Spec}(k)} \mathfrak{X}$  but we will omit the subindex  $\mathfrak{X}$  for convenience. Similarly for all the other algebraic groups  $\mathfrak{G}_I, \mathbf{G}_m$  etc. involved in the proof.) Consider the exact sequence (of sheafs of groups on the flat site of  $\mathfrak{X}$ ; see [18, Expose XXII, 6.2.3])

$$1 \rightarrow \mathfrak{G}_I \rightarrow \mathcal{L}' \rightarrow \mathbf{G}_m^{r'} \rightarrow 1.$$

Since  $\mathcal{L}'$  is split, the above sequence splits and therefore it is exact on the Zariski site of  $\mathfrak{X}$ . Passing to Čech cohomology yields

$$0 \rightarrow H_{\text{Zar}}^1(\mathfrak{X}', \mathfrak{G}_{I'}) \rightarrow H_{\text{Zar}}^1(\mathfrak{X}', \mathcal{L}') \rightarrow H_{\text{Zar}}^1(\mathfrak{X}', \mathbf{G}_m^{r'}) \rightarrow 0.$$

Now if condition (TLT) holds then  $(0) = H_{\text{Zar}}^1(\mathfrak{X}', \mathfrak{T}) = H_{\text{Zar}}^1(\mathfrak{X}', \mathbf{G}_m)^\ell = \text{Pic}(\mathfrak{X}')^\ell$  and therefore  $H_{\text{Zar}}^1(\mathfrak{X}', \mathbf{G}_m^{r'}) = \text{Pic}(\mathfrak{X}')^{r'} = (0)$ . It follows that for establishing (iii), it suffices to show that  $H_{\text{Zar}}^1(\mathfrak{X}', \mathfrak{G}_{I'}) = (0)$ . If now  $\mathcal{L}$  is the standard Levi subgroup of  $\mathfrak{G}$  corresponding to  $I'$ , we can reason as above to conclude that the map  $H_{\text{Zar}}^1(\mathfrak{X}', \mathfrak{G}_{I'}) \rightarrow H_{\text{Zar}}^1(\mathfrak{X}', \mathcal{L})$  has trivial kernel. Since under condition (TLT) we have  $H_{\text{Zar}}^1(\mathfrak{X}', \mathcal{L}) = (0)$  the result follows.  $\square$

**Remark.** No argument is needed for part (iii) of the last proposition in the case of assumption (TRT) (since, unlike (TLT), this assumption does not depend on  $\mathfrak{G}$ ).

**Proposition 10.** Let  $\mathfrak{X}$ ,  $\mathfrak{p}$ , and  $\mathfrak{p}_0$  be as in Proposition 7. Let  $J \triangleleft S(\mathfrak{g}^*)$  be the defining ideal of the closed subset  $\mathfrak{G}(k) \cdot \mathfrak{p}_0 \in \mathfrak{g}$ , and  $\mathcal{L}$  the isotropy group of  $\mathfrak{p}_0$ . Then

- (i) *There exists a canonical isomorphism  $\mathfrak{G}/\mathfrak{L} \simeq \operatorname{Spec}(S(\mathfrak{g}^*)/J)$ .*
- (ii)  *$\mathfrak{p}$  vanishes on  $J$  thereby inducing a scheme morphism  $\psi_{\mathfrak{p}} : \mathfrak{X} \rightarrow \mathfrak{G}/\mathfrak{L}$ .*

**Proof.** The abstract group  $\mathfrak{L}(k)$  acts on the left on  $k[\mathfrak{G}]$  via  $\mathfrak{p} \cdot f(Q) = f(\mathfrak{p}^{-1}Q)$  for all  $\mathfrak{p} \in \mathfrak{L}(k)$ ,  $Q \in \mathfrak{G}(k)$ , and  $f \in k[\mathfrak{G}]$  (where we are identifying  $k[\mathfrak{G}]$  with the ring of polynomial functions of the Zariski closed set corresponding to  $\mathfrak{G}(k)$ ). Since  $\mathfrak{L}$  is reductive the quotient scheme  $\mathfrak{G}/\mathfrak{L}$  exists and it is in fact the affine scheme of the ring of invariants  $B := k[\mathfrak{G}]^{\mathfrak{L}(k)}$  [10, Theorem 1.1]. There is a natural  $k$ -algebra homomorphism  $S(\mathfrak{g}^*) \rightarrow B$  given by  $v \mapsto f_v$  where  $f_v(Q) = v(Q^{-1} \cdot \mathfrak{p}_0)$ . The kernel of this map is the defining ideal  $J \triangleleft S(\mathfrak{g}^*)$  of the closed set  $\mathfrak{G}(k) \cdot \mathfrak{p}_0$ . We thus have an injective  $k$ -algebra homomorphism  $\phi : A \rightarrow B$  where  $A = S(\mathfrak{g}^*)/J$ . The surjectivity of  $\phi$  follows from that of the induced homomorphism  $\bar{\phi} := 1 \otimes \phi : \bar{k} \otimes A \rightarrow \bar{k} \otimes B$ . Now to see that  $\bar{\phi}$  is surjective (in fact an isomorphism) it will suffice to show by [3, 9.1] that  $J$  generates in  $S(\mathfrak{g}(\bar{k})^*)$  the defining ideal of  $\mathfrak{G}(\bar{k}) \cdot \mathfrak{p}_0$ . That  $J$  has this property follows from Proposition 8(ii) applied to  $S = \bar{k}$ .

(ii) We have for all  $x \in \mathfrak{X}$

$$S(\mathfrak{g}^*) \xrightarrow{\mathfrak{p}} R \xrightarrow{\varepsilon_x} k(x).$$

$\mathfrak{p}(x)$

By Propositions 7 and 8(ii)  $\mathfrak{p}(x)$  vanishes on  $J$ , thereby inducing a homomorphism

$$\bar{\mathfrak{p}} : S(\mathfrak{g}^*)/J \simeq k[\mathfrak{G}]^{\mathfrak{L}} \rightarrow R.$$

Indeed if  $f \in J$  then

$$f(\mathfrak{p}) := \mathfrak{p}(f) \in \bigcap_{x \in \mathfrak{X}} x = (0)$$

since  $R$  is reduced. Finally,  $\psi_{\mathfrak{p}}$  is defined to be the scheme morphism corresponding to  $\bar{\mathfrak{p}}$ .  $\square$

**Proposition 11.** *With the notation of Proposition 10 the following are equivalent.*

- (i) *There exists  $\mathfrak{p} \in \mathfrak{G}(R)$  such that  $\mathfrak{p}_0 = \mathfrak{p} \cdot \mathfrak{p}$ .*
- (ii) *There exists a scheme morphism  $\hat{\psi}_{\mathfrak{p}} : \mathfrak{X} \rightarrow \mathfrak{G}$  rendering the diagram*

$$\begin{array}{ccc} & & \mathfrak{G} \\ & \nearrow \hat{\psi}_{\mathfrak{p}} & \downarrow q \\ \mathfrak{X} & \xrightarrow{\psi_{\mathfrak{p}}} & \mathfrak{G}/\mathfrak{L} \end{array}$$

*commutative.*

- (iii) *The pull back  $\operatorname{pr}_1 : \mathfrak{X} \times_{\mathfrak{G}/\mathfrak{L}} \mathfrak{G} \rightarrow \mathfrak{X}$  admits a global section.*

**Proof.** It is well known that (ii) and (iii) are equivalent. To show that (i) and (ii) are equivalent it is best to work in  $k\text{-alg}$ , where by taking Proposition 10 into account the picture is as follows:

$$\begin{array}{ccc}
 & k[\mathfrak{G}] & \\
 \nearrow \mathfrak{p} & \uparrow & \\
 & k[\mathfrak{G}]^{\mathfrak{L}} & \\
 & \wr & \\
 R & \xleftarrow{\bar{\mathfrak{p}}} & S(\mathfrak{g}^*)/J \\
 \nwarrow \mathfrak{p} & \uparrow & \\
 & S(\mathfrak{g}^*) &
 \end{array}$$

Let  $v_1, \dots, v_n$  be a basis of  $\mathfrak{g}$  and  $v^1, \dots, v^n$  the corresponding dual basis of  $\mathfrak{g}^*$ . Then

$$\mathfrak{p} = \sum v_i \otimes \mathfrak{p}(v^i).$$

On the other hand (see the proof of Proposition 10(i)),  $\mathfrak{p}(v^i + J) = v^i(\mathfrak{p}^{-1} \cdot \mathfrak{p}_0)$ . In other words

$$\mathfrak{p}^{-1} \cdot \mathfrak{p}_0 = \sum v_i \otimes \mathfrak{p}(v^i + J).$$

The commutativity of the diagram is thus equivalent to  $\mathfrak{p}$  and  $\mathfrak{p}^{-1} \cdot \mathfrak{p}_0$  being the same element.  $\square$

**Remark 12.** The picture that emerges after the pull back by  $\psi_{\mathfrak{p}}$  is the following:

$$\begin{array}{ccc}
 \mathfrak{X} \times_{\mathfrak{G}/\mathfrak{L}} \mathfrak{G} & \xrightarrow{\text{pr}_2} & \mathfrak{G} \\
 \text{pr}_1 \downarrow & & \downarrow q \\
 \mathfrak{X} & \xrightarrow{\psi_{\mathfrak{p}}} & \mathfrak{G}/\mathfrak{L}
 \end{array}$$

Since the quotient morphism  $q: \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{L}$  is locally trivial (one can see this by means of the big cell, see [18]), the same is the case for the pullback  $\text{pr}_1: \mathfrak{X} \times_{\mathfrak{G}/\mathfrak{L}} \mathfrak{G} \rightarrow \mathfrak{X}$  as a principal homogeneous space for  $\mathfrak{L}$  over  $\mathfrak{X}$ . Condition (TLT) of Theorem 1(iii) ensures that  $\text{pr}_1$  is trivial.

We now turn to the proofs of the last two parts of our main theorem.

**Proof of Theorem 1(ii)(a)** (with  $\mathfrak{X}$  reduced). Let  $\mathfrak{a} \subset \mathfrak{g}(R)$  be a regular MAD. Fix a regular element  $p \in \mathfrak{a}$ . Then for  $p_0$  as in Proposition 11 we have  $\mathfrak{L} = \mathfrak{T}$  where  $\mathfrak{T}$  is the split maximal torus of  $\mathfrak{G}$  corresponding to a fixed split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  containing  $p_0$ . Since  $\mathfrak{T}$  is a product of  $\ell = \text{rank}(\mathfrak{g})$  copies of the multiplicative group  $\mathbf{G}_m = \text{Spec } k[t^{\pm 1}]$ , the  $\mathfrak{T}$ -torsors over  $\mathfrak{X}$  are measured by

$$H_{\text{ét}}^1(\mathfrak{X}, \mathfrak{T}) \simeq H_{\text{ét}}^1(\mathfrak{X}, \mathbf{G}_m)^\ell \simeq \text{Pic}(\mathfrak{X})^\ell$$

([9, Chapter 4, §4] and [6, Chapter 3, §6.3]). The pull-back of Proposition 12(iii) is thus trivial and we conclude that  $p_0 = P \cdot p$  for some  $P \in \mathfrak{G}(R)$ . Then

$$P \cdot \mathfrak{a} \subset P \cdot \mathfrak{z}_{\mathfrak{g}(R)} p = \mathfrak{z}_{\mathfrak{g}(R)} P \cdot p = \mathfrak{z}_{\mathfrak{g}(R)} p_0 = \mathfrak{h}(R).$$

Given that the only  $k$ -diagonalizable elements of  $\mathfrak{h}(R)$  are those of  $\mathfrak{h}$  (see Remark following the proof of Theorem 1(i)) we have  $P \cdot \mathfrak{a} \subset \mathfrak{h}$ , and hence by maximality  $P \cdot \mathfrak{a} = \mathfrak{h}$  as desired.  $\square$

**Proof of Theorem 1(ii)(b)** (with  $\mathfrak{X}$  reduced). By Proposition 11 and Remark 12 if  $p \in \mathfrak{a}$  then  $P \cdot p = p_0$  for some  $P \in \mathfrak{G}(R)$ . We may thus assume with no loss of generality that  $\mathfrak{a} \cap \mathfrak{g} \neq (0)$ . Fix a nonzero element  $p$  in this intersection as well as a split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  with  $p \in \mathfrak{h}$ . We will reason by induction on the rank  $\ell$  of  $\mathfrak{g}$ .

If  $\ell = 1$  then  $\mathfrak{g} \simeq \mathfrak{sl}_2$  so that  $p \neq 0$  amounts to  $p$  being regular and the result holds part (ii)(a). Assume now  $\ell > 1$ .

Let  $V_p \subset \mathfrak{h}^*$  be the  $k$ -span of those  $\alpha \in \Delta = \Delta(\mathfrak{g}, \mathfrak{h})$  satisfying  $\langle \alpha, p \rangle = 0$ . If  $\dim_k V_p = 0$  then  $p$  is regular. We may thus assume  $0 < \dim_k V_p < \ell$ . Let  $\Delta_p = \Delta \cap V_p$ . As mentioned in Proposition 9,  $\Delta_p$  is a root system on  $V_p$  and there exists a base  $\Pi = \{\alpha_1 \dots \alpha_\ell\}$  of  $\Delta$  and a subset  $I \subset \{1, \dots, \ell\}$  such that  $\Pi_I := \{\alpha_i : i \in I\}$  is a base of  $\Delta_p$ .

With such a  $\Pi$  and  $I$  fixed let  $\mathfrak{s}$  be the subalgebra of  $\mathfrak{g}$  generated by the  $\mathfrak{g}^{\pm \alpha_i}$ 's with  $i \in I$ , and let  $\mathfrak{r}$  be the subalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and  $\mathfrak{s}$ . Then  $\mathfrak{r}$  is reductive with  $\mathfrak{s}$  as semisimple part, and centre  $\mathfrak{c}$  given by the  $k$ -span of the coweights  $\omega_j^\vee$ ,  $j \notin I$ . Note also that  $\mathfrak{r} = \mathfrak{z}_{\mathfrak{g}}(p)$  and therefore that

$$\mathfrak{r}(R) = \mathfrak{s}(R) \oplus \mathfrak{c}(R),$$

where  $\mathfrak{c}(R)$  is the centre and  $\mathfrak{s}(R)$  the derived algebra of  $\mathfrak{r}(R)$ . Since  $p \in \mathfrak{a}$  and  $\mathfrak{a}$  is abelian we also have  $\mathfrak{r}(R) = \mathfrak{z}_{\mathfrak{g}(R)}(p) \supset \mathfrak{a}$ .

Let

$$\mathfrak{b} = \{s \in \mathfrak{s}(R) : c + s \in \mathfrak{a} \text{ some } c \in \mathfrak{c}(R)\}.$$

Then  $\mathfrak{b} = \pi(\mathfrak{a})$  where  $\pi : \mathfrak{r}(R) \rightarrow \mathfrak{s}(R)$  is the canonical homomorphism. In particular,  $\mathfrak{a} \subset \mathfrak{c}(R) + \mathfrak{b}$  and  $\mathfrak{b}$  is an abelian  $k$ -diagonalizable subalgebra of  $\mathfrak{s}(R)$ .

Now  $\mathfrak{b}$  is contained in a MAD of  $\mathfrak{s}(R)$ . By induction together with Proposition 9(ii), we then deduce the existence of an element  $\mathfrak{p} \in \mathfrak{G}_I(R)$  such that  $\mathfrak{p} \cdot \mathfrak{b} \subset \mathfrak{h}_{\mathfrak{p}} := \sum_{\alpha \in \Delta_{\mathfrak{p}}} [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]$ . Since the elements of  $\mathfrak{G}_I(R)$  fix  $\mathfrak{c}(R)$  pointwise we get

$$\mathfrak{p} \cdot \mathfrak{a} \subset \mathfrak{p} \cdot (\mathfrak{c}(R) + \mathfrak{b}) \subset \mathfrak{c}(R) + \mathfrak{h} \subset \mathfrak{h}(R).$$

As before given that elements of  $\mathfrak{p} \cdot \mathfrak{a}$  are  $k$ -diagonalizable we in fact have  $\mathfrak{p} \cdot \mathfrak{a} \subset \mathfrak{h}$ , and hence by maximality that  $\mathfrak{p} \cdot \mathfrak{a} = \mathfrak{h}$ .  $\square$

**End of the proof of Theorem 1.** Let  $\mathcal{J}$  be the nilradical of  $R$  and let  $R' = R/\mathcal{J}$ . Then  $\mathfrak{X}_{\text{red}} = \text{Spec}(R')$  is connected, has a rational point, and by assumption satisfies property (TLT). It follows then from Proposition 11 and Remark 12 that if  $\mathfrak{p}'$  denotes the natural image of  $\mathfrak{p}$  in  $\mathfrak{g}(R')$ , then  $\mathfrak{p}' \cdot \mathfrak{p}' = \mathfrak{p}'(x_0 + \mathcal{J}) = \mathfrak{p}_0 \in \mathfrak{g} \subset \mathfrak{g}(R')$  for some  $\mathfrak{p}' \in \mathfrak{G}(R')$ . We claim that there exists  $\mathfrak{p} \in \mathfrak{G}(R)$  lifting  $\mathfrak{p}'$  and such that  $\mathfrak{p} \cdot \mathfrak{p} = \mathfrak{p}_0$ . This will finish the proof since we can then reason as in the proofs of the reduced case above.

To establish the claim we may assume that  $R$  is noetherian. In this case  $\mathcal{J}$  is nilpotent and by considering  $\mathcal{J} \supset \mathcal{J}^2 \supset \mathcal{J}^4 \supset \dots \supset \mathcal{J}^{2^n} = (0)$  it will suffice to establish the claim under the assumption  $\mathcal{J}^2 = (0)$ . Since  $\mathfrak{G}$  is smooth it then follows that  $\mathfrak{p}'$  does lift to an element  $\mathfrak{p}_1$  of  $\mathfrak{G}(R)$ . Thus

$$\mathfrak{p}_1 \cdot \mathfrak{p} = \mathfrak{p}_0 + \sum_{i=1}^s \alpha_i^{\vee} \otimes \varepsilon_i + \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} v_{\alpha} \otimes \varepsilon_{\alpha},$$

where  $\{\alpha_i^{\vee}, v_{\alpha}\}$  is a Chevalley basis of  $\mathfrak{g}$  and the  $\varepsilon_i$ 's and  $\varepsilon_{\alpha}$ 's belong to  $\mathcal{J}$ .

For  $\alpha \notin \Delta_0 := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : \langle \alpha, \mathfrak{p}_0 \rangle = 0\}$  let  $\theta_{\alpha} = \exp(\text{ad}(v_{\alpha} \otimes \langle \alpha, \mathfrak{p}_0 \rangle^{-1} \varepsilon_{\alpha}))$ . This is an automorphism of  $\mathfrak{g}(R)$  that can be realized as the adjoint action of an element  $\mathfrak{p}_{\alpha}$  of  $\mathfrak{G}(R)$  [6, II, §6.3.7]. If we now set  $\mathfrak{p}_2 = \prod_{\alpha \notin \Delta_0} \mathfrak{p}_{\alpha}$  (the product taken in any order) and  $\mathfrak{p} = \mathfrak{p}_2 \mathfrak{p}_1 \in \mathfrak{G}(R)$  we have  $\mathfrak{p} \cdot \mathfrak{p} = \mathfrak{p}_0 + \mathfrak{q}$  where  $\mathfrak{q} = \sum \alpha_i^{\vee} \otimes \varepsilon_i + \sum_{\alpha \in \Delta_0} v_{\alpha} \otimes \varepsilon_{\alpha}$ . Since  $\mathfrak{p} \cdot \mathfrak{p}$  is  $k$ -diagonalizable and commutes with  $\mathfrak{p}_0$  it follows that  $\text{ad } \mathfrak{q}$  is  $k$ -diagonalizable. On the other hand  $\text{ad } \mathfrak{q}$  is visibly nilpotent. Thus  $\mathfrak{q} = 0$  and  $\mathfrak{p} \cdot \mathfrak{p} = \mathfrak{p}_0$  as desired.  $\square$

### 13. An interesting example

We look at  $\mathfrak{g} = \mathfrak{sl}_2$  and  $\mathfrak{X} = \mathfrak{G}/\mathfrak{T}$  (the “generic” regular case for  $\mathfrak{sl}_2$ ). Here the group  $\mathfrak{G}$  is  $\mathbf{SL}_2$  and  $R = k[\mathfrak{G}]^{\mathfrak{T}(k)}$ . For convenience we will denote  $k[\mathfrak{G}]$  by  $S$ . Consider the element  $\text{id} \in \mathfrak{G}(S)$ . Let

$$\begin{aligned} \mathfrak{p} := \text{id}^{-1} \cdot h &= \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \\ &= \begin{pmatrix} x_{11}x_{22} + x_{12}x_{21} & 2x_{12}x_{22} \\ -2x_{11}x_{21} & -(x_{11}x_{22} + x_{12}x_{21}) \end{pmatrix}. \end{aligned}$$

Observe that  $\mathfrak{p} \in \mathfrak{g}(R)$ . Furthermore  $\mathfrak{p}$  is regular (since  $h$  is) and  $k$ -diagonalizable (since  $\text{ad}_{\mathfrak{g}(S)} \mathfrak{p}$  is and  $\text{ad}_{\mathfrak{g}(R)} \mathfrak{p}$  is simply its restriction to  $\mathfrak{g}(R)$ ).

Say  $\mathfrak{q} \in \mathfrak{g}(R)$  is such that  $[\mathfrak{p}, \mathfrak{q}] = 2\mathfrak{q}$ . Again by looking inside  $\mathfrak{g}(S)$  we see that

$$\mathfrak{q} = \text{id}^{-1} \cdot \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} sx_{21}x_{22} & sx_{22}^2 \\ sx_{21}^2 & sx_{21}x_{22} \end{pmatrix}$$

for some  $s \in S$ .

Observe that, save for the  $s$ , all entries of  $\mathfrak{q}$  belong to  $S^\alpha$ . It follows that  $\mathfrak{q} \in \mathfrak{g}(R)$  only if  $s \in S^{-\alpha}$ . Since the  $R$ -module  $S^{-\alpha}$  is (rank one projective but) not free we conclude that  $\text{ad}_{\mathfrak{g}(R)} \mathfrak{p}$  is  $k$ -diagonalizable as a  $k$ -linear but *not* as an  $R$ -linear endomorphism of  $\mathfrak{g}(R)$ .

Next we look at the  $k$ -algebra homomorphism  $\bar{\mathfrak{p}}$  attached to  $\mathfrak{p}$  described in Proposition 10. Let  $\{E, H, F\}$  be the basis of  $\mathfrak{g}^*$  dual to  $\{e, h, f\}$ . Identify  $S(\mathfrak{g}^*)$  with the polynomial ring  $k[E, H, F]$ . Then  $\mathfrak{p} : S(\mathfrak{g}^*) \rightarrow R$  is given by

$$\mathfrak{p} = \begin{cases} E \mapsto 2x_{12}x_{22}, \\ H \mapsto x_{11}x_{22} + x_{12}x_{21}, \\ F \mapsto -2x_{11}x_{21}. \end{cases}$$

Our present situation is depicted by the diagram

$$\begin{array}{ccc} & & S \\ & & \uparrow \\ & & R \\ & & \downarrow \wr \\ R & \xleftarrow{\bar{\mathfrak{p}}} & S(\mathfrak{g}^*)/J \\ & \nwarrow \mathfrak{p} & \uparrow \\ & & S(\mathfrak{g}^*) \end{array}$$

and we can identify  $\bar{\mathfrak{p}}$  with an endomorphism of  $R$ . Our choice of  $x_0 \in \mathfrak{X}$  is the maximal ideal of  $R$  obtained by intersecting  $R$  with the ideal of  $S$  generated by  $x_{11} - 1$ ,  $x_{12}$ ,  $x_{21}$ , and  $x_{22} - 1$ . Then  $\mathfrak{p}_0 = h$  and under our isomorphism to  $E + J \in S(\mathfrak{g}^*)/J$  corresponds  $f_E \in R$  with  $f_E(\text{id}) = E(\text{id}^{-1} \cdot \mathfrak{p}_0) = 2x_{12}x_{22}$ . Similar considerations apply to  $H$  and  $F$  thus showing that the endomorphism  $\bar{\mathfrak{p}}$  we are after is in fact the identity map.

According to Proposition 11 then, conjugacy is equivalent to the principal  $\mathfrak{T}$ -bundle  $q : \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{T}$  being trivial. This however is not the case as the bundle in question is a generator of  $\text{Pic}(\mathfrak{X}) \simeq \mathbb{Z}$ . Note that  $S/R$  is fppf and that our bundle becomes trivial over  $S$  as one can see directly (since by construction  $\mathfrak{p}$  is conjugate to  $\mathfrak{p}_0$  under  $\mathfrak{G}(S)$ ), or abstractly (since  $\text{Pic}(\mathfrak{G})$  is trivial).

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